

Equations in dual variables for Whittaker functions.

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Abstract

It is known that the Whittaker functions $w(q, \lambda)$ associated to the group $SL(N)$ are eigenfunctions of the Hamiltonians of the open Toda chain, hence satisfy a set of differential equations in the Toda variables q_i . Using the expression of the q_i for the closed Toda chain in terms of Sklyanin variables λ_i , and the known relations between the open and the closed Toda chains, we show that Whittaker functions also satisfy a set of new difference equations in λ_i .

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1 Introduction.

The method of separation of variables is acquiring a central place in the domain of integrable systems. In the classical case, the separated variables are simply the poles of the eigenvectors of the Lax matrix. Together with the spectral curve, they are the necessary data to reconstruct the eigenvectors from their analyticity properties, and therefore the Lax matrix itself. The solution of the integrable system is given the fact that the image of the divisor of poles by the Abel map evolves linearly on the Jacobian under an integrable flow.

In the quantum case, these separated variables were first defined and used by Sklyanin. In particular, he showed that the original N -body Schrödinger equation separates into N one dimensional equations, called Baxter equations. The separated variables also appear in the N -soliton form factor formulae in sine-Gordon theory. It was shown that Smirnov formulae have a simple interpretation in terms of the separated variables of the N solitons [7].

More recently, it was realized that the separated variables provided a convenient approach to the problem of quantizing a classical integrable system. It is interesting that this idea emerged independently both among mathematicians [11] and physicists [12].

In [14], we took this point of view to solve the quantum inverse problem for the closed Toda chain. In this note, we remark that this result immediately suggests that the Whittaker functions associated to $Sl(N)$, which are the eigenfunctions of the open Toda chain, see eqs.(11) below, should also satisfy a set of equations in the dual (momentum) variables, eqs.(18) below. We prove these relations by using the Mellin-Barnes integral representation for the Whittaker functions found in [10].

2 The Closed Toda Chain.

The closed Toda chain is defined by the Hamiltonian

$$H = \sum_{i=1}^{N+1} \frac{1}{2} p_i^2 + e^{q_{i+1}-q_i} \quad (1)$$

where we assume that $q_{N+2} \equiv q_1$, and canonical Poisson brackets

$$\{q_i, q_j\} = 0, \quad \{p_i, q_j\} = \delta_{ij}, \quad \{p_i, p_j\} = 0$$

As it is well known, one can associate to this system a 2×2 Lax matrix as follows. Consider the matrices

$$T_j(\lambda) = \begin{pmatrix} \lambda + p_j & -e^{q_j} \\ e^{-q_j} & 0 \end{pmatrix}$$

and construct

$$T(\lambda) = T_1(\lambda) \cdots T_2(\lambda) T_{N+1}(\lambda) \quad (2)$$

We can write

$$T(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}; \quad A(\lambda)D(\lambda) - B(\lambda)C(\lambda) = 1 \quad (3)$$

$A(\lambda)$ is a polynomial of degree $N + 1$, $D(\lambda)$ is of degree $N - 1$, and $B(\lambda)$, $C(\lambda)$ are of degree n . The spectral curve is defined as usual

$$\det(T(\lambda) - \mu) = 0 \quad \equiv \quad \mu + \mu^{-1} - t(\lambda) = 0 \quad (4)$$

where

$$t(\lambda) = A(\lambda) + D(\lambda) = \lambda^{N+1} + \sum_{j=0}^N \lambda^j H_j, \quad H_N = P, \quad H_{N-1} = \frac{1}{2}P^2 - H$$

where $P = \sum_i p_i$, and H is given by eq.(1). We will consider the system reduced by the translational symmetry. We fix the total momentum $P = 0$. The symplectic form $\omega = \sum_{i=1}^{N+1} dp_i \wedge dq_i$ becomes $\omega_{reduced} = \sum_{i=1}^N dp_i \wedge d(q_i - q_{N+1})$ so that the canonical coordinates of the reduced system can be taken as $(p_i, q_i - q_{N+1})$, $i = 1 \cdots N$. We choose the gauge condition $q_{N+1} = 0$. It must be emphasized that this reduced system, which does not contain the degree of freedom p_{N+1}, q_{N+1} anymore, is not the open Toda chain. The N quantities H_j are conserved and Poisson commute.

The separated variables are the poles of the eigenvectors of the Lax matrix. For a 2×2 matrix of the form eq.(3) the eigenvector is simple

$$(T(\lambda) - \mu)\Psi = 0, \quad \Psi = \begin{pmatrix} 1 \\ \psi_2 \end{pmatrix}, \quad \psi_2 = -\frac{A(\lambda) - \mu}{B(\lambda)}$$

The poles of ψ_2 at finite distance are above the zeroes λ_i of $B(\lambda) = 0$ which is a polynomial of degree n . The two points above λ_i are $\mu_i^+ = A(\lambda_i)$, $\mu_i^- = D(\lambda_i)$ so that ψ_2 has a pole only at the second point. The points of the dynamical divisor are therefore $(\lambda_i, D(\lambda_i))$, $B(\lambda_i) = 0$.

The Inverse Problem consists in reconstructing everything in terms of the (λ_i, μ_i) . Its solution for the quantum theory was given in [14].

Quantum commutation relations are defined directly on the separated variables.

$$[\lambda_k, \lambda_{k'}] = 0, \quad \mu_k \lambda_{k'} = (\lambda_{k'} + i\hbar \delta_{kk'}) \mu_k, \quad [\mu_k, \mu_{k'}] = 0$$

To reconstruct the Hamiltonians themselves is simple. The n points λ_i, μ_i^- belonging to the spectral curve, we have the equations (we drop the superscript $-$ in μ_i^-):

$$\mu_i + \mu_i^{-1} - \lambda_i^{N+1} - \sum_{j=1}^N \lambda_i^j H_j = 0$$

This is a linear system of N equations for the N quantities H_j . Its solution makes sense even quantum mechanically and, quite generally, produces commuting Hamiltonians [11, 12].

To reconstruct the original operators p_i, q_i , we need the operators $X^{(k)}, Y^{(k)}$ below. We call $[k]$ a subset of cardinality k of $(1, 2, \dots, N)$: $[k] = (i_1, i_2, \dots, i_k)$. We write $\sum_{[k]}$ for the sum over all such subsets. Let

$$S_{[k]}(\lambda) = \prod_{i \in [k]} \prod_{j \notin [k]} \frac{1}{(\lambda_i - \lambda_j)}, \quad \text{and} \quad \mu_{[k]} = \mu_{i_1} \mu_{i_2} \cdots \mu_{i_k} \quad (5)$$

and define

$$X^{(k)} \equiv \sum_{[k]} S_{[k]}(\lambda) \mu_{[k]} \quad (6)$$

$$Y^{(k)} \equiv \sum_{[k]} S_{[k]}(\lambda) \left(\sum_{i \notin [k]} \lambda_i \right) \mu_{[k]} \quad (7)$$

The quantum Toda operators are given by:

$$e^{q_i} = (-1)^{N-1} \frac{X^{(N-i+1)}}{X^{(N-i)}}, \quad p_i = \frac{Y^{(N-i+1)}}{X^{(N-i+1)}} - \frac{Y^{(N-i)}}{X^{(N-i)}} \quad (8)$$

The canonical commutation relations

$$[e^{q_i}, e^{q_j}] = 0, \quad [e^{q_i}, p_j] = -i\hbar \delta_{ij} e^{q_i}, \quad [p_i, p_j] = 0 \quad (9)$$

are a simple consequences of the following quadratic algebra:

$$\begin{aligned} [X^{(k)}, X^{(l)}] &= 0 \\ [X^{(k)}, Y^{(l)}] &= i\hbar(k-l)\theta(k-l)X^{(k)}X^{(l)} \\ [Y^{(k)}, Y^{(l)}] &= i\hbar(k-l)\left[\theta(k-l)Y^{(k)}X^{(l)} + \theta(l-k)Y^{(l)}X^{(k)}\right] \end{aligned}$$

Note that this implies that there is no ordering ambiguity in the expressions eq.(8).

The operators $X^{(k)}$ are self adjoint with respect to the non trivial scalar product

$$(f, g) = \int_{-\infty}^{\infty} d\lambda \, m(\lambda) \, f^*(\lambda) g(\lambda)$$

where

$$m(\lambda) = \prod_{i < j} \frac{1}{\left| \Gamma\left(\frac{\lambda_i - \lambda_j}{i\hbar}\right) \right|^2} \quad (10)$$

3 Whittaker vectors and functions.

There is an important connection between the closed and open Toda chains that we now recall.

The matrix element $B(\lambda)$ in eq.(3) can be written as

$$B(\lambda) = -e^{q_{N+1}} \sum_{k=0}^N \lambda^{N-k} h_k(p, q)$$

where $h_k(p, q)$ are the Hamiltonians of the *open* Toda chain obtained by removing particle $N+1$ from the closed chain [2].

The eigenfunctions of the Hamiltonians $h_k(p, q)$ satisfy

$$h_k(p, q) w(\lambda, q) = \sigma_k(\lambda) w(\lambda, q) \quad (11)$$

where $\sigma_k(\lambda)$ are the symmetric functions of the λ_i . It is a beautiful result by Kostant and Semenov-Tian-Shansky that $w(\lambda, q)$ is the Whittaker function of $sl(N)$. Thus the function $w(\lambda, q)$ is the kernel of the Fourier transform going from the q to the λ (momentum) variables. They satisfy the completeness relation

$$\int d\lambda m(\lambda) w^*(q, \lambda) w(q', \lambda) = \delta(q - q') \quad (12)$$

where $m(\lambda)$ is given by eq.(10), and the orthogonality relations [9]

$$\int dq w^*(q, \lambda) w(q, \lambda') = m^{-1}(\lambda) \delta(\lambda - \lambda') \quad (13)$$

To recall how such functions arise, consider the $sl(N)$ Lie algebra

$$[H_j, E_{\pm\alpha_i}] = \pm a_{ij} E_{\pm\alpha_i}, \quad [E_{\alpha_i}, E_{-\alpha_j}] = \delta_{ij} H_i, \quad i, j = 1, \dots, N-1$$

where α_i are the simple root vectors and a_{ij} is the Cartan matrix. The Weyl vector is

$$\rho = \frac{1}{2} \sum_{\alpha > 0} \alpha$$

The quadratic Casimir operator is

$$C_2 = \sum_{\alpha} E_{\alpha} E_{-\alpha} + \sum_{i,j} a_{ij}^{-1} H_i H_j$$

where the \sum_{α} runs over all positive and negative roots.

Whittaker vectors are such that

$$E_{\alpha_i} |w_{\lambda}\rangle = \mu_{\alpha_i}^R |w_{\lambda}\rangle, \quad \alpha_i \text{ simple} \quad (14)$$

Of course, if α is a non-simple root, then $E_{\alpha} |w_{\lambda}\rangle = 0$. Similarly, dual Whittaker vectors satisfy

$$\langle w_{\lambda} | E_{-\alpha_i} = \mu_{\alpha_i}^L \langle w_{\lambda} |, \quad \alpha_i \text{ simple} \quad (15)$$

We may furthermore assume that $|w_{\lambda}\rangle$ belongs to some irreducible representation with weight λ . In that case, we have

$$C_2 |w_{\lambda}\rangle = c_2(\lambda) |w_{\lambda}\rangle$$

Define the Whittaker function

$$w(\lambda, q) = e^{\rho(q)} \langle w_{\lambda} | e^q | w_{\lambda} \rangle$$

where $q = \sum_{ij} q_i a_{ij}^{-1} H_j$ belongs to the Cartan subalgebra. Then, we have

$$\begin{aligned} c_2(\lambda) w(\lambda, q) &= e^{\rho(q)} \langle w_{\lambda} | e^q C_2 | w_{\lambda} \rangle \\ &= e^{\rho(q)} \langle w_{\lambda} | e^q \left(2 \sum_{\alpha > 0} E_{-\alpha} E_{\alpha} + 2H_{\rho} + \sum_{i,j} a_{ij}^{-1} H_i H_j \right) | w_{\lambda} \rangle \end{aligned}$$

Expanding $H_{\alpha} = \sum_i (\alpha)_i a_{ij}^{-1} H_j$, where a_{ij} is the Cartan matrix, we get

$$c_2(\lambda) w(\lambda, q) = \left(a_{ij} \frac{\partial^2}{\partial q_i \partial q_j} + 2 \sum_{\alpha \text{ simple}} \mu_{\alpha}^L \mu_{\alpha}^R e^{-\alpha(q)} \right) w(\lambda, q)$$

The differential operator in the right hand side is just the Hamiltonian of the open Toda chain. Hence $w(\lambda, q)$ satisfy to the open Toda chain Schroedinger equation! Clearly, the same analysis can be done for higher order Casimirs giving rise to the higher order Hamiltonians of the open chain.

There exists several integral representations for the Whittaker functions.

- **The "Gauss" representation** [6].

$$w_G(\lambda, q) = \int dz_{ij} \prod_{i=1}^{N-1} \Delta_i^{-(\lambda, \alpha_i)-1} (zS^{-1}) e^{\sum_i \mu_i^R e^{\alpha_i(q)} z_{i,i+1} + \mu_{N-i}^L \frac{\Delta_{i,i+1}(zS^{-1})}{\Delta_i(zS^{-1})}}$$

The integral is over upper-triangular matrices $N \times N$ matrices, z , with 1 on the diagonal. S is the antidiagonal matrix $S_{ij} = \delta_{N+1-i,j}$. $\Delta_i(M)$ is the determinant of the $i \times i$ submatrix of M consisting in the first i rows and columns. $\Delta_{i-1,i}(M)$ is defined as the determinant of the $(i-1) \times (i-1)$ submatrix of M with columns $i-1$ and i interchanged.

- **The "Iwasawa" representation** [6].

$$w_I(\lambda, q) = e^{\lambda(q)} \int dz_{ij} \prod_{i=1}^{N-1} \Delta_i^{-\frac{1}{2}(\lambda, \alpha_i) - \frac{1}{2}} (zz^t) e^{i \sum_i \mu_i z_{i,i+1} e^{\alpha_i(q)}}$$

The definitions of z and $\Delta_i(M)$ are the same as above.

- **The "Mellin-Barnes" representation** [10].

The Weyl invariant Whittaker function has a representation in terms of multiple Mellin-Barnes integrals. Let γ_{jk} be a lower triangular $N \times N$ matrix.

$$\begin{pmatrix} \gamma_{11} & 0 & \cdots & & \\ \gamma_{21} & \gamma_{22} & 0 & & \cdots \\ \vdots & & & & \\ \gamma_{N-1,1} & \gamma_{N-1,2} & \cdots & \gamma_{N-1,N-1} & 0 \\ \gamma_{N1} & \gamma_{N2} & \cdots & \cdots & \gamma_{NN} \end{pmatrix}$$

We identify

$$\gamma_{Nj} = \lambda_j, \quad j = 1 \cdots N$$

Then

$$w(\lambda, q) = \int \prod_{i,j=1}^{N-1} d\gamma_{ij} e^{(-\frac{1}{i\hbar} \sum_{n,k=1}^N q_n (\gamma_{nk} - \gamma_{n-1,k}))} \times \prod_{n=1}^{N-1} \left\{ \frac{\prod_{j=1}^n \prod_{k=1}^{n+1} \hbar^{\frac{\gamma_{nj} - \gamma_{n+1,k}}{i\hbar}} \Gamma(\frac{\gamma_{nj} - \gamma_{n+1,k}}{i\hbar})}{\prod_{j < k=1}^n |\Gamma(\frac{\gamma_{nj} - \gamma_{n,k}}{i\hbar})|^2} \right\} \quad (16)$$

The integration contour is as follows.

$$\begin{aligned} \text{Im } \gamma_{11} &> \max(\text{Im } \gamma_{12}, \text{Im } \gamma_{22}) \\ \text{Im } \gamma_{21}, \text{Im } \gamma_{22} &> \max(\text{Im } \gamma_{31}, \text{Im } \gamma_{32}, \text{Im } \gamma_{33}) \\ &\vdots \\ \text{Im } \gamma_{N-1,1}, \cdots, \text{Im } \gamma_{N-1,N-1} &> \max(\text{Im } \gamma_{N1}, \cdots, \text{Im } \gamma_{NN}) \end{aligned}$$

Note that the poles of the integrand are located at

$$\gamma_{nj} = \gamma_{n+1,k} - i\hbar s, \quad s \in N$$

so we can move the contour γ_{nj} upward safely.

4 Diagonalization of $X^{(k)}$.

Since the operators $X^{(k)}$ are all commuting, we can diagonalize them simultaneously. The $X^{(k)}$ are self-adjoint with respect to the measure $m(\lambda)$ eq.(10), their eigenfunctions satisfy the same completeness and orthogonality conditions as the Whittaker functions diagonalizing the Hamiltonians of the open Toda chain eqs.(12,13).

The Whittaker functions are the kernel of the Fourier transform going from the variables q_i to the variables λ_i . Similarly, the eigenfunctions of the operators $X^{(k)}$ are the kernel of the Fourier transform going from the variables λ_i to the variables q_i . So, it is natural to expect

$$X^{(k)}(\lambda, \mu) w^*(\lambda, q) = (-1)^{k(N-1)} e^{\sum_{i=1}^k q_{N+1-i}} w^*(\lambda, q) \quad (17)$$

Taking the complex conjugate of this equation, we get

$$X^{(k)*}(\lambda, \mu) w(\lambda, q) = (-1)^{k(N-1)} e^{\sum_{i=1}^k q_{N+1-i}} w(\lambda, q) \quad (18)$$

where $X^{(k)*}$ is the complex conjugate of $X^{(k)}$, i.e. it is given by eq.(6) with μ_i replaced by μ_i^* the shift operator $\lambda_i \rightarrow \lambda_i - i\hbar$. Note that eqs.(11) are differential equations in q_i , while eqs.(18) are difference equations in λ_i .

In the remaining of this section, we prove that the function $w(\lambda, q)$ defined by the Mellin-Barnes integral representation eq.(16), which is known to satisfy eqs.(11), also satisfy eqs.(18).

Before treating the general case, it is instructive to do the calculation for $N = 2$ and $N = 3$ first.

4.1 The case $N = 2$.

For $N = 2$, the function $w(\lambda, q)$ reads:

$$w(\lambda, q) = \int d\gamma_{11} e^{-\frac{1}{i\hbar} [q_1 \gamma_{11} + q_2 (\lambda_1 - \gamma_{11} + \lambda_2)]} \prod_{k=1}^2 (i\hbar)^{\frac{\gamma_{11} - \lambda_k}{i\hbar}} \Gamma\left(\frac{\gamma_{11} - \lambda_k}{i\hbar}\right)$$

The operators $X^{(k)*}$ are

$$X^{(1)*} = \frac{1}{\lambda_1 - \lambda_2} (\mu_1^* - \mu_2^*), \quad X^{(2)*} = \mu_1^* \mu_2^*$$

We have

$$\begin{aligned}
X^{(1)*} w(\lambda, q) &= \\
&= \frac{i\hbar e^{q_2}}{\lambda_{12}} \int d\gamma_{11} e^{-\frac{1}{i\hbar}[q_1\gamma_{11}+q_2(\lambda_1-\gamma_{11}+\lambda_2)]} \prod_{k=1}^2 (i\hbar)^{\frac{\gamma_{11}-\lambda_k}{i\hbar}} \Gamma\left(\frac{\gamma_{11}-\lambda_k}{i\hbar}\right) \left[\frac{(\gamma_{11}-\lambda_1)}{i\hbar} - \frac{(\gamma_{11}-\lambda_2)}{i\hbar} \right] = \\
&= -e^{q_2} \int d\gamma_{11} e^{-\frac{1}{i\hbar}[q_1\gamma_{11}+q_2(\lambda_1-\gamma_{11}+\lambda_2)]} \prod_{k=1}^2 (i\hbar)^{\frac{\gamma_{11}-\lambda_k}{i\hbar}} \Gamma\left(\frac{\gamma_{11}-\lambda_k}{i\hbar}\right)
\end{aligned}$$

Hence, we have proved

$$X^{(1)*} w(\lambda, q) = -e^{q_2} w(\lambda, q)$$

Similarly

$$\begin{aligned}
X^{(2)*} w(\lambda, q) &= \int d\gamma_{11} e^{-\frac{1}{i\hbar}[q_1\gamma_{11}+q_2(\lambda_1-\gamma_{11}+\lambda_2-2i\hbar)]} \prod_{k=1}^2 (i\hbar)^{\frac{\gamma_{11}-\lambda_k+i\hbar}{i\hbar}} \Gamma\left(\frac{\gamma_{11}-\lambda_k+i\hbar}{i\hbar}\right) = \\
&= e^{q_2+q_1} \int d\gamma'_{11} e^{-\frac{1}{i\hbar}[q_1\gamma'_{11}+q_2(\lambda_1-\gamma'_{11}+\lambda_2)]} \prod_{k=1}^2 (i\hbar)^{\frac{\gamma'_{11}-\lambda_k}{i\hbar}} \Gamma\left(\frac{\gamma'_{11}-\lambda_k}{i\hbar}\right)
\end{aligned}$$

where $\gamma'_{11} = \gamma_{11} + i\hbar$. So, the integral on the right hand side is the same as the initial one but on a contour shifted upward by $+i\hbar$. But as we already noticed, this does not change the value of the integral. Hence, we have proved

$$X^{(2)*} w(\lambda, q) = e^{q_1+q_2} w(\lambda, q)$$

4.2 The case $N = 3$.

For $N = 3$, we have:

$$\begin{aligned}
w(\lambda, q) &= \int d\gamma e^{-\frac{1}{i\hbar}[q_1\gamma_{11}+q_2(\gamma_{21}-\gamma_{11}+\gamma_{22})+q_3(\lambda_1+\lambda_2+\lambda_3-\gamma_{21}-\gamma_{22})]} \times \\
&\times \frac{\prod_{j,k} (i\hbar)^{\frac{\gamma_{1j}-\gamma_{2k}}{i\hbar}} \Gamma\left(\frac{\gamma_{1j}-\gamma_{2k}}{i\hbar}\right)}{\prod_{j < k} |\Gamma\left(\frac{\gamma_{1j}-\gamma_{1k}}{i\hbar}\right)|^2} \times \frac{(i\hbar)^{\frac{\gamma_{2j}-\lambda_k}{i\hbar}} \Gamma\left(\frac{\gamma_{2j}-\lambda_k}{i\hbar}\right)}{\prod_{j < k} |\Gamma\left(\frac{\gamma_{2j}-\gamma_{2k}}{i\hbar}\right)|^2}
\end{aligned} \tag{19}$$

The operators $X^{(k)*}$ read

$$\begin{aligned}
X^{(1)*} &= \frac{1}{\lambda_{12}\lambda_{13}} \mu_1^* + \frac{1}{\lambda_{21}\lambda_{23}} \mu_2^* + \frac{1}{\lambda_{31}\lambda_{32}} \mu_3^* \\
X^{(2)*} &= \frac{1}{\lambda_{13}\lambda_{23}} \mu_1^* \mu_2^* + \frac{1}{\lambda_{12}\lambda_{32}} \mu_1^* \mu_3^* + \frac{1}{\lambda_{21}\lambda_{31}} \mu_2^* \mu_3^* \\
X^{(3)*} &= \mu_1^* \mu_2^* \mu_3^*
\end{aligned}$$

Consider first $X^{(1)*} w(\lambda, q)$. The exponential factor in the first line of eq.(19) produces the factor e^{q_3} . In the integrand, using $\Gamma(x+1) = x\Gamma(x)$, we get the factor

$$\frac{(i\hbar)^2}{\lambda_{12}\lambda_{13}} \frac{(\gamma_{21}-\lambda_1)}{i\hbar} \frac{(\gamma_{22}-\lambda_1)}{i\hbar} + \frac{(i\hbar)^2}{\lambda_{21}\lambda_{23}} \frac{(\gamma_{21}-\lambda_2)}{i\hbar} \frac{(\gamma_{22}-\lambda_2)}{i\hbar} + \frac{(i\hbar)^2}{\lambda_{31}\lambda_{32}} \frac{(\gamma_{21}-\lambda_3)}{i\hbar} \frac{(\gamma_{22}-\lambda_3)}{i\hbar}$$

which is equal to 1. Hence, we have proved

$$X^{(1)*}w(\lambda, q) = e^{q_3}w(\lambda, q)$$

Next, we consider $X^{(2)*}w(\lambda, q)$. The exponential factor produces the factor e^{2q_3} . Using the Γ function relation, we get in the integrand

$$\begin{aligned} & \frac{1}{\lambda_{13}\lambda_{23}}(\gamma_{21} - \lambda_1)(\gamma_{22} - \lambda_1)(\gamma_{21} - \lambda_2)(\gamma_{22} - \lambda_2) + \\ & \frac{1}{\lambda_{12}\lambda_{32}}(\gamma_{21} - \lambda_1)(\gamma_{22} - \lambda_1)(\gamma_{21} - \lambda_3)(\gamma_{22} - \lambda_3) + \\ & \frac{1}{\lambda_{21}\lambda_{31}}(\gamma_{21} - \lambda_2)(\gamma_{22} - \lambda_2)(\gamma_{21} - \lambda_3)(\gamma_{22} - \lambda_3) \end{aligned}$$

which is equal to

$$\frac{(\gamma_{21} - \lambda_1)(\gamma_{21} - \lambda_2)(\gamma_{21} - \lambda_3) - (\gamma_{22} - \lambda_1)(\gamma_{22} - \lambda_2)(\gamma_{22} - \lambda_3)}{(\gamma_{21} - \gamma_{22})}$$

So, we get a sum of two identical integrals but with γ_{21} and γ_{22} interchanged. Let us treat the first one. Using the Γ functions, we reconstruct

$$\Gamma\left(\frac{\gamma_{11} - \gamma_{2k}}{i\hbar}\right) \times \frac{\prod_k \Gamma\left(\frac{\gamma_{21} - \lambda_k}{i\hbar} + 1\right) \Gamma\left(\frac{\gamma_{22} - \lambda_k}{i\hbar}\right)}{\Gamma\left(\frac{\gamma_{21} - \gamma_{22}}{i\hbar} + 1\right) \Gamma\left(\frac{\gamma_{22} - \gamma_{21}}{i\hbar}\right)}$$

We now change variables $\gamma_{21} = \gamma'_{21} - i\hbar$. The exponential produces the factor $e^{q_2 - q_3}$. The integrand becomes

$$\Gamma\left(\frac{\gamma_{11} - \gamma_{21}}{i\hbar}\right) \Gamma\left(\frac{\gamma_{11} - \gamma_{22}}{i\hbar}\right) \times \frac{\prod_k \Gamma\left(\frac{\gamma_{21} - \lambda_k}{i\hbar}\right) \Gamma\left(\frac{\gamma_{22} - \lambda_k}{i\hbar}\right)}{\Gamma\left(\frac{\gamma_{21} - \gamma_{22}}{i\hbar}\right) \Gamma\left(\frac{\gamma_{22} - \gamma_{21}}{i\hbar}\right)} \times \frac{\gamma_{11} - \gamma_{21}}{\gamma_{22} - \gamma_{21}}$$

Note that there is no pole at $\gamma_{21} = \gamma_{11}$ nor at $\gamma_{21} = \gamma_{22}$, so that we can move back the γ_{12} contour to its original position. To this, we have to add the same expression with γ_{21} and γ_{22} interchanged. We reproduce $w(\lambda, q)$ because

$$\frac{\gamma_{11} - \gamma_{21}}{\gamma_{22} - \gamma_{21}} + \frac{\gamma_{11} - \gamma_{22}}{\gamma_{21} - \gamma_{22}} = 1$$

Thus we have proved

$$X^{(2)*}w(\lambda, q) = e^{q_2 + q_3}w(\lambda, q)$$

Finally, we consider $X^{(3)*}w(\lambda, q)$. The exponential factor produces the factor e^{3q_3} . Then we change variables $\gamma_{2j} + i\hbar = \gamma'_{2j}$, $\gamma_{11} + i\hbar = \gamma'_{11}$, which amounts to shifting the contours by $i\hbar$ upward. This produces a factor $e^{q_1 + q_2 - 2q_3}$. So we have proved that

$$X^{(3)*}w(\lambda, q) = e^{q_1 + q_2 + q_3}w(\lambda, q)$$

4.3 General case.

We have

$$X^{(k)*}w(\lambda, q) = \sum_{[k]} S_{[k]} \mu_{[k]}^* w(\lambda, q)$$

The exponential in the first line of eq.(16) produces a factor e^{kq_N} . In the integrand we get the factor

$$\sum_{[k]} S_{[k]}(\gamma_N) \prod_{j \in [k]} \prod_{i=1}^{N-1} (\gamma_{N-1,i} - \gamma_{Nj})$$

By eq.(20) in the Appendix, this is equal to

$$(-1)^{(N-k)k} \sum_{[k-1]} S_{[k-1]}(\gamma_{N-1}) \prod_{i \in [k-1]} \prod_{j=1}^N (\gamma_{N-1,i} - \gamma_{Nj})$$

Let us keep track of $\gamma_{N-1,i}, i \in [k-1]$. The factors $\prod_{i \in [k-1]} \prod_{j=1}^N (\gamma_{N-1,i} - \gamma_{Nj})$ are absorbed into the Γ -functions in the numerator of the integrand, while the factor $S_{[k-1]}(\gamma_{N-1})$ is absorbed in the Γ -functions in the denominator of the integrand, to produce

$$(-1)^{(N-k)k} \prod_{i \in [k-1]} \frac{\prod_{j=1}^N \Gamma\left(\frac{\gamma_{N-1,i} - \gamma_{Nj}}{i\hbar} + 1\right)}{\prod_{j \notin [k-1]} \Gamma\left(\frac{\gamma_{N-1,i} - \gamma_{N-1,j}}{i\hbar} + 1\right) \Gamma\left(\frac{\gamma_{N-1,j} - \gamma_{N-1,i}}{i\hbar}\right)}$$

Next, we change variables $\gamma_{N-1,i} + i\hbar = \gamma'_{N-1,i}, i \in [k-1]$. The exponential factor in the first line of eq.(16) is symmetrical in all the $\gamma_{N-1,i}$ and yields

$$e^{(k-1)(q_{N-1} - q_N)}$$

Then, the Γ -functions with shifted arguments are

$$(-1)^{k(N-k)} \prod_{i \in [k-1]} \frac{\prod_{j=1}^{N-2} \Gamma\left(\frac{\gamma_{N-2,j} - \gamma_{N-1,i}}{i\hbar} + 1\right)}{\prod_{j \notin [k-1]} \Gamma\left(\frac{\gamma_{N-1,j} - \gamma_{N-1,i}}{i\hbar} + 1\right)}$$

which produce a factor

$$(-1)^{(N-k)} S_{[k-1]}(\gamma_{N-1}) \prod_{i \in [k-1]} \prod_{j=1}^{N-2} (\gamma_{N-2,j} - \gamma_{N-1,i})$$

in the integrand. So, we are back to the same problem, but at level $k-1, N-1$. We eventually reach the level $1, N-k+1$, where we use the identity

$$\sum_i \prod_{j \neq i} \frac{1}{\gamma_{N-k+1,i} - \gamma_{N-k+1,j}} \prod_{j=1}^{N-k} (\gamma_{N-k,j} - \gamma_{N-k+1,i}) = (-1)^{(N-k)}$$

Putting everything together, we arrive at

$$X^{(k)*}w(\lambda, q) = (-1)^{k(N-1)} e^{q_N + q_{N-1} + \dots + q_{N-k+1}} w(\lambda, q)$$

which is exactly eq.(18).

5 Conclusion.

To conclude, we would like to mention that the operators $X^{(k)}$ are limiting cases of the Ruijsenaars-Macdonald operators:

$$M^{(k)} = \sum_{[k]} \prod_{\substack{i \in [k] \\ j \notin [k]}} \frac{tq^{\lambda_i} - t^{-1}q^{\lambda_j}}{q^{\lambda_i} - q^{\lambda_j}} \mu_{[k]}$$

So, our result is probably a limiting case of the results obtained in [13].

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6 Appendix.

Proposition 1 *One has the identity*

$$\sum_{[k]} S_{[k]}(\gamma_N) \prod_{j \in [k]} \prod_{i=1}^{N-1} (\gamma_{N-1,i} - \gamma_{Nj}) = (-1)^{(N-k)k} \sum_{[k-1]} S_{[k-1]}(\gamma_{N-1}) \prod_{i \in [k-1]} \prod_{j=1}^N (\gamma_{N-1,i} - \gamma_{Nj}) \quad (20)$$

Proof. First of all, the left-hand side is a polynomial in γ_N . It has potential poles at $\gamma_{Ni} = \gamma_{Nj}$. Let us suppose $i = 1, j = 2$. One has to assume that $1 \in [k], 2 \notin [k]$ or vice versa, otherwise there is no pole. In the above sum, we consider the two terms

$$[k] = 1 + [k'], \quad [k] = 2 + [k'], \quad 1, 2 \notin [k']$$

Denote by $[l']$ the complementary subset of $[k']$ in $3, 4, \dots, N$. The two terms are, respectively

$$\begin{aligned} & \frac{1}{\gamma_{N1} - \gamma_{N2}} \prod_{j \in [l']} \frac{1}{\gamma_{N1} - \gamma_{Nj}} \prod_{i \in [k']} \frac{1}{\gamma_{Ni} - \gamma_{N2}} \prod_{\substack{i \in [k'] \\ j \in [l']}} \frac{1}{\gamma_{Ni} - \gamma_{Nj}} \prod_{j=1}^{N-1} (\gamma_{N-1,j} - \gamma_{N1}) \prod_{i \in [k']} (\gamma_{N-1,j} - \gamma_{Ni}) \\ & \frac{1}{\gamma_{N2} - \gamma_{N1}} \prod_{j \in [l']} \frac{1}{\gamma_{N2} - \gamma_{Nj}} \prod_{i \in [k']} \frac{1}{\gamma_{Ni} - \gamma_{N1}} \prod_{\substack{i \in [k'] \\ j \in [l']}} \frac{1}{\gamma_{Ni} - \gamma_{Nj}} \prod_{j=1}^{N-1} (\gamma_{N-1,j} - \gamma_{N2}) \prod_{i \in [k']} (\gamma_{N-1,j} - \gamma_{Ni}) \end{aligned}$$

It follows that the sum of the two residues cancel. From the behaviour at ∞ , we see that both sides are polynomials of degree $k - 1$. To show that they are identical, we compare the values at the $N - 1$ points $\gamma_{N1} = \gamma_{N-1,i}$. It is enough to consider

$$\gamma_{N1} = \gamma_{N-1,1}$$

In the left hand side, only the sets $[k]$ such that $1 \notin [k]$ contribute. Hence we get

$$\sum_{1 \notin [k]} \prod_{i \in [k]} \frac{1}{\gamma_{Ni} - \gamma_{N1}} \prod_{\substack{i \in [k] \\ j \notin [k], j \neq 1}} \frac{1}{\gamma_{Ni} - \gamma_{Nj}} \prod_{j \in [k]} (\gamma_{N-1,1} - \gamma_{Nj}) \prod_{i=2}^{N-1} (\gamma_{N-1,i} - \gamma_{Nj})$$

When evaluated at $\gamma_{N1} = \gamma_{N-1,1}$, the first and third terms cancel and we are left with

$$(-1)^k \sum_{1 \notin [k]} \prod_{\substack{i \in [k] \\ j \notin [k], j \neq 1}} \frac{1}{\gamma_{Ni} - \gamma_{Nj}} \prod_{i=2}^{N-1} \prod_{j \in [k]} (\gamma_{N-1,i} - \gamma_{Nj})$$

In the right hand side, only the sets $[k-1]$ such that $1 \notin [k-1]$ contribute. Hence, we get

$$(-1)^{(N-k)k} \sum_{[k-1] \in [k-1]} \prod_{i \in [k-1]} \frac{1}{\gamma_{N-1,i} - \gamma_{N-1,1}} \prod_{\substack{i \in [k-1] \\ j \neq 1}} \frac{1}{\gamma_{N-1,i} - \gamma_{N-1,j}} \prod_{i \in [k-1]} (\gamma_{N-1,i} - \gamma_{N1}) \prod_{j=2}^N (\gamma_{N-1,i} - \gamma_{Nj})$$

When evaluated at $\gamma_{N1} = \gamma_{N-1,1}$, the first and third terms cancel and we are left with

$$(-1)^{(N-k)k} \sum_{1 \notin [k-1]} \prod_{\substack{i \in [k-1] \\ j \neq 1}} \frac{1}{\gamma_{N-1,i} - \gamma_{N-1,j}} \prod_{i \in [k-1]} \prod_{j=2}^N (\gamma_{N-1,i} - \gamma_{Nj})$$

The two things are identical if our identity holds at level $N-1$. The lowest level is $N=k$. There, we have

$$S_{[N]}(\gamma_N) = 1, \quad S_{[N-1]}(\gamma_{N-1}) = 1$$

and the identity reduces to

$$\prod_{j=1}^N \prod_{i=1}^{N-1} (\gamma_{N-1,i} - \gamma_{Nj}) = \prod_{i=1}^{N-1} \prod_{j=1}^N (\gamma_{N-1,i} - \gamma_{Nj})$$

which is obviously true. ■

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